Single-Phase Flow through Porous Media

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The local volume average of the equation of motion is taken for an incompressible fluid flowing through a porous structure under conditions such that inertial effects may be neglected. The result has two terms beyond a pressure gradient: g, the force per unit volume which a flowing fluid exerts on a porous structure, and the divergence of the local volume-averaged extra stress tensor (viscous portion of the stress tensor).

Constitutive equations for g are examined with the aid of the principle of material indifference. When g is assumed to be a function of the velocity of the fluid relative to the solid as well as various scalars, the usual results for a nonoriented (isotropic) porous structure are obtained. When g is assumed to be a function of the local porosity gradient as well, we derive a new expression for g applicable to oriented (anisotropic) porous structures.

For a Newtonian fluid with a constant viscosity, the divergence of the local volume-averaged extra stress tensor is proportional to the Laplacian of the averaged velocity vector. Boundary conditions for the averaged velocity vector are discussed. Three problems are solved for the flow of an incompressible Newtonian fluid in a nonoriented permeable medium. These solutions, as well as an order-of-magnitude analysis, suggest that we may often neglect both the Laplacian of average velocity and the boundary conditions for the tangential components of averaged velocity at an impermeable wall.

Two specific constitutive equations for g are proposed for the flow of incompressible Noll simple fluids in nonoriented porous structures. Flow through a porous medium bounded by an impermeable cylindrical surface is solved for these two constitutive equations, and the results are compared with previously available experimental data.

In a prior paper (*1*), we introduced a method for local-averaging of the equations of continuity and of motion. We defined a resistance transformation which in part maps the local average velocity vector into the local force per unit volume which the fluid exerts on the pore walls. For an isotropic porous structure the proper values of the resistance transformation were all equal and termed the resistance coefficient; the functional dependence of this resistance coefficient was discussed using the Buckingham-Pi theorem for both Newtonian and viscoelastic fluids.

The results of our local volume-averaging of the equations of continuity and of motion (*1*) can be readily summarized. Let us consider a particular point in a porous medium and let there be associated with this point a closed surface S, the volume of which is V. We may assume that this point lies in the interior of S, but this is not essential. This same surface S may be associated with every point in the porous medium by a translation without rotation. For example, if S is a unit sphere, the center of which coincides with the point considered, we may center on each point in the porous medium a unit sphere. Let $V_{(s)}$ denote the pores which contain fluid in the interior of S; the volume and shape of $V_{(s)}$ in general will change from point to point in the porous medium. The closed boundary surface of $V_{(s)}$, $S_{(s)}$, is the sum of S_e and S_f: S_e coincides with S and S_f with the pore walls. The local volume-averages of the equations of continuity and of motion are respectively

$$\frac{\partial \bar{\rho}}{\partial t} + \text{div}\,(\overline{\rho\ \mathbf{v}}) = 0 \tag{1}$$

and

$$\frac{\partial\, \overline{\rho\ \mathbf{v}}}{\partial\, t} + \text{div}\ \overline{(\rho\ \mathbf{v}\ \mathbf{v})} = \text{div}\,\overline{\mathbf{T}} + \overline{\rho\ \mathbf{f}} + \frac{1}{V}\int_{S_f} \mathbf{T}\cdot\mathbf{n}\ dS \tag{2}$$

The overbar indicates a local volume-averaged variable. If B is any scalar, vector, or tensor, we define

$$\overline{B} \equiv \frac{1}{V}\int_{V_{(s)}} B\ dV \tag{3}$$

In words, $\overline{B}$ is an average over V of a quantity B associated with the fluid.

In this discussion we limit ourselves to an incompressible fluid, for which Equation (1) reduces to

$$\text{div}\ \overline{\mathbf{v}} = 0 \tag{4}$$

We assume that all inertial effects may be neglected in the local volume-averaged equation of motion (*1, 2*) and that the external force per unit mass f may be represented by a scalar potential ϕ,

$$\mathbf{f} = -\nabla\phi \tag{5}$$

Equation (2) may consequently be written as

$$\nabla\,(\overline{\mathcal{P} - p_0}) - \text{div}(\overline{\mathbf{T} + p\ \mathbf{I}}) - \frac{1}{V}\int_{S_f} [\mathbf{T} + (p_0 - \rho\phi)\ \mathbf{I}\,]\cdot\mathbf{n}\ dS = 0 \tag{6}$$

where we define

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$$\overline{\mathcal{P}} \equiv \overline{p} + \rho\, \overline{\phi} \quad (7)$$

A constant reference or ambient pressure p_0 is introduced here in order that we may identify g,

$$\mathbf{g} \equiv -\frac{1}{V} \int_{S_f} [\mathbf{T} + (p_0 - \rho\, \phi)\, \mathbf{I}] \cdot \mathbf{n}\, dS \quad (8)$$

as the force per unit volume which the fluid exerts on the pore walls contained within S beyond the hydrostatic force and beyond any force attributable to the ambient pressure. This force per unit volume g is entirely assignable to the motion of the fluid.

In (*1*) we introduced a resistance transformation **K** by requiring that

$$-\operatorname{div}\,(\overline{\mathbf{T} + p\ \mathbf{I}}) + \mathbf{g} = \mathbf{K} \cdot \overline{\mathbf{v}} \quad (9)$$

It is more natural to treat the two terms on the left of this equation separately, which we do in the next two sections. After discussing several example problems for incompressible Newtonian fluids, we suggest that for many situations the first term on the left of Equation (9) may be neglected with respect to g. We conclude with a treatment of one-dimensional viscoelastic flow and a comparison with experimental data.

FORCE EXERTED UPON THE SOLID BY THE FLUID

For the moment let us assume that g is a function of the difference between the local average velocity of the fluid, $\overline{\mathbf{v}}$, and the local average velocity of the solid, $\overline{\mathbf{u}}$:

$$\mathbf{g} = \hat{\mathbf{g}}(\overline{\mathbf{v}} - \overline{\mathbf{u}}) \quad (10)$$

Functional dependence upon several scalars such as porosity, a characteristic length of the porous medium, a characteristic viscosity of the fluid, and a characteristic time of the fluid is understood in this expression. As the result of this explicit dependence upon both a characteristic viscosity and a characteristic time of the fluid, this development is designed to apply both to Newtonian fluids and to viscoelastic fluids such as the incompressible Noll simple fluid (*3, 4*). We define

$$\overline{\mathbf{u}} = \frac{1}{V} \int_{V - V_{(s)}} \mathbf{v}\, dV \quad (11)$$

A change of frame is a one-to-one mapping of space-time onto itself such that distances, time intervals, and a sense of time are preserved. The most general change of frame is of the form (*5*, p. 22):

$$\mathbf{p}^* = \mathbf{c}\,(t) + \mathbf{Q}\,(t) \cdot \{\mathbf{p} - \mathbf{p}_0\} \quad (12)$$

$$t^* = t - a \quad (13)$$

Here $\mathbf{p}^*$ and $\mathbf{p}$ are the position vector fields in the new and old frames respectively; $\mathbf{p}_0$ is a fixed position in the old frame; a is a constant. We may think of $\mathbf{c}(t)$ as a time-dependent reference position or origin, since $\mathbf{p}_0$ is mapped into $\mathbf{c}(t)$. The time-dependent orthogonal transformation $\mathbf{Q}(t)$ represents a rotation and, possibly, a reflection from a right- (left) handed coordinate system to a left- (right) handed one.

A scalar is said to be *frame indifferent* (*5*, p. 22), if it does not change its value under a change of frame. A vector **b** is said to be frame indifferent, if

$$\mathbf{b} = \mathbf{p}_1 - \mathbf{p}_2 \quad (14)$$

implies that

$$\mathbf{b}^* = \mathbf{p}_1^* - \mathbf{p}_2^* \quad (15)$$

By Equation (12), we have

$$\mathbf{b} = \mathbf{Q} \cdot \mathbf{b} \quad (16)$$

Forces are regarded here as being given *a priori* and consequently as being frame indifferent (*5*, p. 27). In particular,

$$\mathbf{g}^* = \mathbf{Q} \cdot \mathbf{g} \quad (17)$$

Equation (10) is a constitutive equation in the sense that it describes the behavior of the fluid in the porous medium. The principle of frame indifference (*5*, p. 34; *6*, p. 44) says that constitutive equations should be indifferent under a change of reference. Another way of saying this is that any two observers should come to the same conclusions regarding the behavior of a material in a particular dynamical process. This means that

$$\mathbf{Q}(t) \cdot \hat{\mathbf{g}}(\overline{\mathbf{v}} - \overline{\mathbf{u}}) = \hat{\mathbf{g}}(\overline{\mathbf{v}}^* - \mathbf{u}^*)$$
$$= \hat{\mathbf{g}}(\mathbf{Q}(t) \cdot \{\overline{\mathbf{v}} - \overline{\mathbf{u}}\}) \quad (18)$$

We have used here the fact that, while velocity is not frame-indifferent (*5*, p. 23), a difference of velocities is indifferent when they are referred to the same frame. By Equation (18), $\hat{\mathbf{g}}$ is an isotropic function (*6*, p. 22) and, by a representation theorem for a vector-valued isotropic function of one vector (*6*, p. 35), we may write

$$\mathbf{g} = \hat{\mathbf{g}}(\overline{\mathbf{v}} - \overline{\mathbf{u}}) = R\{\overline{\mathbf{v}} - \overline{\mathbf{u}}\} \quad (19)$$

It is to be understood here that the resistance coefficient R is a function of the magnitude of the local volume-averaged velocity of the fluid relative to the local volume-averaged velocity of the solid, $|\overline{\mathbf{v}} - \overline{\mathbf{u}}|$, as well as the various scalars mentioned previously. The only difference between the treatment for a Newtonian fluid and that for a viscoelastic fluid lies in the functional dependence of R (*1*, see "Some simple models . . .").

One would not expect Equation (19) to be valid for a porous structure in which porosity, ψ, is a function of position. For such a structure Equation (10) must be altered to include a dependence upon additional vector and possibly tensor quantities. For example, one might postulate a dependence of g upon the local porosity gradient as well as upon $\overline{\mathbf{v}} - \overline{\mathbf{u}}$,

$$\mathbf{g} = \hat{\mathbf{g}}(\overline{\mathbf{v}} - \overline{\mathbf{u}},\ \nabla\psi) \quad (20)$$

The principle of material frame indifference again requires $\hat{\mathbf{g}}$ to be an isotropic function:

$$\mathbf{Q}(t) \cdot \hat{\mathbf{g}}(\overline{\mathbf{v}} - \overline{\mathbf{u}},\ \nabla\psi)$$
$$= \hat{\mathbf{g}}\,[\mathbf{Q}(t) \cdot \{\overline{\mathbf{v}} - \overline{\mathbf{u}}\},\ \mathbf{Q}(t) \cdot \nabla\psi] \quad (21)$$

By representation theorems of Spencer and Rivlin (*7*, section 7) and of Smith (*8*), the most general polynomial isotropic vector function of two vectors is of the form

$$\mathbf{g} = \phi_{(1)} \{ \overline{\mathbf{v}} - \overline{\mathbf{u}} \} + \phi_{(2)} \nabla \psi \tag{22}$$

where $\phi_{(1)}$ and $\phi_{(2)}$ are scalar-valued polynomials in $|\overline{\mathbf{v}} - \overline{\mathbf{u}}|$, $|\nabla\psi|$, and $[\{\overline{\mathbf{v}} - \overline{\mathbf{u}}\} \cdot \nabla\psi]$. [In applying the theorem of Spencer and Rivlin, we identify a vector $\mathbf{b}$ which has covariant components b_i with the skew-symmetric tensor which has contravariant components $\epsilon^{ijk}\, b_i$. Their theorem requires an additional term in Equation (22) proportional to the vector product $\{\overline{\mathbf{v}} - \overline{\mathbf{u}}\} \wedge \nabla \psi$. This term is not consistent with the requirement that $\hat{g}$ be isotropic (*6*, p. 24) and consequently is dropped.] We expect $\phi_{(2)} = 0$ for $|\overline{\mathbf{v}} - \overline{\mathbf{u}}| = 0$, in order that $\mathbf{g} = 0$ in this limit.

A so-called "anisotropic" porous structure is one for which flow behavior depends upon a direction (or a set of directions) intrinsically associated with the pore geometry, such as $\nabla\psi$ in Equation (20). This seems to be an unfortunate use of the word anisotropic, since $\hat{g}$ in Equation (20) is an isotropic function by the principle of material frame indifference. For this reason, we hereafter refer to Equations (10) and (19) as describing nonoriented porous structures; we say that Equations (20) and (22) describe oriented porous media.

DIVERGENCE OF THE EXTRA STRESS

We limit ourselves at this point to an incompressible Newtonian fluid and assume that viscosity is a constant, at least locally in $V_{(s)}$:

$$\overline{(\mathbf{T} + p\ \mathbf{I})} = \mu[\overline{\nabla \mathbf{v}} + \overline{(\nabla \mathbf{v})^t}] \tag{23}$$

Previously (*1*), we proved that

$$\nabla \int_{V_{(s)}} B\ dV = \int_{S_e} B\ \mathbf{n}\ dS \tag{24}$$

where B may be a scalar, vector, or tensor. Since the velocity of the fluid is zero at the pore walls, S_f, we have

$$\begin{aligned} \overline{\nabla \mathbf{v}} &= \frac{1}{V} \int_{V_{(s)}} \nabla \mathbf{v}\ dV = \frac{1}{V} \int_{S_e + S_f} \mathbf{v}\ \mathbf{n}\ dS \\ &= \frac{1}{V} \int_{S_e} \mathbf{v}\ \mathbf{n}\ dS = \nabla \overline{\mathbf{v}} \end{aligned} \tag{25}$$

In arriving at the second step of Equation (25), we make use of the generalized Green's transformation. Exactly the same argument may be used to show that

$$\overline{(\nabla \mathbf{v})^t} = (\nabla \overline{\mathbf{v}})^t \tag{26}$$

By Equations (23), (25), and (26) we find that

$$\overline{(\mathbf{T} + p\ \mathbf{I})} = \mu[\nabla \overline{\mathbf{v}} + (\nabla \overline{\mathbf{v}})^t] \tag{27}$$

Equations (4) and (27) allow us to conclude that

$$\operatorname{div} \overline{(\mathbf{T} + p\ \mathbf{I})} = \mu \operatorname{div} (\nabla \overline{\mathbf{v}}) \tag{28}$$

MODIFIED DARCY'S LAW AND SUITABLE BOUNDARY CONDITIONS

With Equations (8), (19), and (26), Equation (6) becomes for an incompressible Newtonian fluid moving through a stationary nonoriented (see definition under *Force exerted . . .*) porous structure:

$$\nabla \overline{(\mathcal{P} - p_0)} - \mu \operatorname{div} (\nabla \overline{\mathbf{v}}) + R\ \overline{\mathbf{v}} = 0 \tag{29}$$

This may be regarded as a modified form of Darcy's law for a Newtonian fluid. [A similar equation was proposed without derivation by Brinkman (*9*).] Comparing this with equation (22) of (*1*), we see that the second term on the left of Equation (29) is new. The boundary conditions which we require the local volume-averaged velocity field to satisfy have a bearing on the importance of this term. Since this is a second-order partial differential equation in $\overline{\mathbf{v}}$, we should expect to satisfy more boundary conditions on $\overline{\mathbf{v}}$ than are satisfied using Darcy's law.

Consider flow through a porous structure which is bounded by impermeable walls. We might be thinking of flow through a bed of sand packed in a pipe or through a layer of sandstone bounded by relatively impermeable granite. As we advance into the pipe wall or granite from the porous structure, the averaged velocity of the fluid, $\overline{\mathbf{v}}$, decreases, since less of the averaging surface S intersects the porous structure which contains the fluid. At some distance ϵ inside the impermeable wall, $\overline{\mathbf{v}}$ becomes zero (this distance ϵ will depend upon the averaging surface chosen). This is a boundary condition which must be satisfied by the average velocity field $\overline{\mathbf{v}}$.

One might ask whether $\overline{\mathbf{v}}$ also goes to zero just inside each grain boundary of the porous medium. It will generally be convenient to choose S so large that g need no longer be thought of as an explicit function of position (although it may be an implicit function of position through its dependence on $\overline{\mathbf{v}}$, ψ, . . .). Such an S will intersect many pores, precluding the necessity that $\overline{\mathbf{v}}$ be zero at points inside the solid particles which compose the porous structure.

We visualize the average velocity field to be one-dimensional in flow through a bed of sand packed in a tube. Yet a solution of this form cannot satisfy both the usual form of Darcy's law for a nonoriented porous structure [*1* equation (22)] and a boundary condition which requires all components of the average velocity vector to be zero on some cylindrical surface. A primary advantage of Equation (29) is that a one-dimensional solution for this geometry can satisfy such a boundary condition.

In what follows we solve several simple problems of practical interest in order to gain an appreciation for the importance of this boundary condition for the average velocity vector near an impermeable wall and consequently for the importance of the second term on the left side of Equation (29).

Channel flow

We wish to analyze in rectangular Cartesian coordinates flow of an incompressible Newtonian fluid through a nonoriented permeable structure of uniform porosity bounded by two infinite parallel planes. We take as boundary conditions on the averaged velocity vector and the averaged pressure

$$\text{at} \quad y = b + \epsilon: \quad \overline{\mathbf{v}} = 0 \tag{30}$$

$$\text{at} \quad y = -b - \epsilon: \quad \overline{\mathbf{v}} = 0 \tag{31}$$

$$\text{at} \quad x = 0, \quad y = b, \quad z = 0: \quad \overline{p} = \psi\, P_0 \tag{32}$$

and

$$\text{at} \quad x = L, \quad y = b, \quad z = 0: \quad \overline{p} = \psi\, P_L \tag{33}$$

In Equations (32) and (33), we recognize that experimentalists measure more nearly $<p> = \overline{p}/\psi$ [see *1*, Equation (21)].

Let us assume that there is a solution of the form

$$\overline{v}_x = \overline{v}_x(y), \ \overline{v}_y = \overline{v}_z = 0 \tag{34}$$

The equation of continuity, Equation (4), is satisfied by this form of averaged velocity field. The three components of Equation (29) reduce to

$$\frac{\partial(\overline{\mathcal{P}} - p_0)}{\partial x} - \mu \frac{d^2 \overline{v}_x}{dy^2} + R\ \overline{v}_x = 0 \tag{35}$$

and

$$\frac{\partial(\overline{\mathcal{P}} - p_0)}{\partial y} = \frac{\partial(\overline{\mathcal{P}} - p_0)}{\partial z} = 0 \tag{36}$$

These equations imply that

$$C = \frac{d(\overline{\mathcal{P}} - p_0)}{dx} = \mu \frac{d^2 \overline{v}_x}{dy^2} - R\ \overline{v}_x = \text{a constant} \tag{37}$$

We can integrate the first step of Equation (37) with boundary conditions (32) and (33) to obtain

$$C = -\frac{1}{L} [\psi\{P_0 - P_L\} + \rho\{\overline{\phi}(0, b, 0) - \overline{\phi}(L, b, 0)\}] \tag{38}$$

where $\overline{\phi} = \overline{\phi}(x, y, z)$

When we integrate the second step of Equation (37) with boundary conditions (30) and (31), we have

$$\overline{v}_x = -\frac{C}{R} \left[1 - \frac{\cosh(N\ y^*)}{\cosh\ (N)} \right] \tag{39}$$

where we define

$$N = \sqrt{\frac{R}{\mu}}\ (b + \epsilon), \quad y^* = \frac{y}{b + \epsilon} \tag{40}$$

If we take μ/R to be of at least the same order of magnitude as the usual permeability k [compare with Equation (1) of (*1*)], we see that N would ordinarily be a very large number for flow through a rock structure. For a 250 millidarcy rock we would estimate $\sqrt{R/\mu}$ to be approximately 2×10^4 cm.$^{-1}$. This would mean that from Equation (39) $\overline{v}_x$ would be relatively independent of y except in the immediate vicinity of the walls.

It is often convenient to choose the averaging surface S to include the entire cross section available for flow. When this is done, it is only the center-line velocity which is of interest. Under these circumstances we see that from Equation (39)

$$\overline{v}_x|_{y=0} \doteq -\frac{C}{R} \tag{41}$$

the usual result for this geometry.

Tube flow

Flow through a porous structure enclosed by an infinite impermeable tube is a similar problem. As boundary conditions we take

$$\text{at} \quad r = r_0 + \epsilon: \quad \overline{\mathbf{v}} = 0 \tag{42}$$

$$\text{at} \quad r = r_0, \quad \theta = 0, \quad z = 0: \quad \overline{p} = \psi\, P_0 \tag{43}$$

and

$$\text{at} \quad r = r_0, \quad \theta = 0, \quad z = L: \quad \overline{p} = \psi\, P_L \tag{44}$$

If we seek a solution of the form

$$\overline{v}_r = \overline{v}_\theta = 0, \quad \overline{v}_z = \overline{v}_z(r) \tag{45}$$

we find

$$\overline{v}_z = -\frac{C}{R} \left[1 - \frac{I_0(N\ r^*)}{I_0(N)} \right] \tag{46}$$

where $\overline{\phi} = \overline{\phi}(r, \theta, z)$,

$$C = -\frac{1}{L} [\psi\{P_0 - P_L\} + \rho\{\overline{\phi}(r_0, 0, 0) - \overline{\phi}(r_0, 0, L)\}] \tag{47}$$

and

$$N = \sqrt{\frac{R}{\mu}}\ (r_0 + \epsilon), \quad r^* = \frac{r}{r_0 + \epsilon} \tag{48}$$

By I_0 we mean the zero-order modified Bessel function of the first kind (*10*, p. 143). For most porous structures, $\overline{v}_z$ would be essentially constant over the cross section of the tube except in the immediate vicinity of the wall where it approaches zero.

If one is interested only in the center-line velocity because the averaging surface S has been chosen to include the entire cross section for flow, we have the common result:

$$\overline{v}_z|_{r=0} \doteq -\frac{C}{R} \tag{49}$$

Radial flow

A problem closely related to radial flow to or from a well bore may be described in cylindrical coordinates by a velocity distribution of the form

$$\overline{v}_r = \overline{v}_r(r), \quad \overline{v}_\theta = \overline{v}_z = 0 \tag{50}$$

with an associated averaged pressure distribution which satisfies

$$\text{at} \quad r = r_1, \quad \theta = 0, \quad z = 0: \quad \overline{p} = \psi\, P_1 \tag{51}$$

and

$$\text{at} \quad r = r_2, \quad \theta = 0, \quad z = 0: \quad \overline{p} = \psi\, P_2 \tag{52}$$

We assume here that the cylindrical coordinate system is oriented with the z axis in the direction opposite to the action of gravity. Let us ask whether there is such a solution to Equations (4) and (29).

From Equation (4), we have

$$\frac{1}{r} \frac{d}{dr} (r\ \overline{v}_r) = 0 \tag{53}$$

which implies that

$$\overline{v}_r = \frac{C}{r} \tag{54}$$

where C is a constant. The three components of Equation (29) reduce to

$$\frac{\partial(\overline{\mathcal{P}} - p_0)}{\partial r} + R\ \overline{v}_r = 0 \tag{55}$$

and

$$\frac{\partial(\overline{\mathcal{P}} - p_0)}{\partial \theta} = \frac{\partial(\overline{\mathcal{P}} - p_0)}{\partial z} = 0 \tag{56}$$

or

$$\frac{d(\overline{\mathcal{P}} - p_0)}{dr} = \frac{d\overline{p}}{dr} = -\frac{RC}{r} \tag{57}$$

Upon integration of this last with boundary conditions (51) and (52), we obtain

$$C = \frac{\psi\{P_1 - P_2\}}{R\ \ln(r_2/r_1)} \tag{58}$$

Equations (54) and (58) imply that

$$\overline{v}_r = \frac{\psi\{P_1 - P_2\}}{r\ R\ \ln(r_2/r_1)} \qquad (59)$$

This is the same result as one gets from a more standard approach [starting with Equation (22) of (*1*)].

IMPORTANCE OF THE LAPLACIAN OF VELOCITY IN EQUATION (29)

Let us put Equation (29) in a dimensionless form:

$$\frac{P}{L\ R\ v}\ \nabla^+(\overline{\mathcal{P} - p_0})^+ - \frac{\mu}{R\ L^2}\ \nabla^+\overline{\mathbf{v}}^+ + \overline{\mathbf{v}}^+ = 0 \quad (60)$$

Here P is a characteristic pressure, v is a characteristic magnitude of velocity, and L is a length characteristic of the gross geometry. If, as we suggested earlier, μ/R is of the same order of magnitude as usual permeability, then $\mu/(RL^2)$ would be a very small number for common porous media problems. This suggests that the second term on the left of Equation (29) could be neglected with respect to the third term. But the reader is cautioned that this is an intuitive argument which need not be true in every situation. There is no theorem which says that because a term in a differential equation is multiplied by a small parameter it can be neglected.

Nevertheless, this argument appears to be a helpful one in the problems just examined. So long as one is not concerned with the averaged velocity distribution in the immediate vicinity of a solid boundary, it does not appear important to satisfy boundary conditions on the tangential components of volume-averaged velocity or to include the effects of the Laplacian of the averaged velocity in Equation (29).

As an example of the type of problem where one might get into trouble using the above argument, consider flow through a porous-walled tube. The description of the averaged velocity distribution in the immediate neighborhood of the boundary of the porous medium would appear to be important in determining the proper boundary conditions for the fluid flowing through the tube. Since the tangential component of velocity would not necessarily go to zero at the tube wall, one might expect to see in experimental data an apparent slip at the tube wall.

VISCOELASTIC FLUIDS

Essentially the same argument as indicated above can be made for an incompressible Noll simple fluid (*5*, p. 64; *6*, p. 81 and 427), a very general description of the behavior of incompressible viscoelastic fluids. An incompressible Noll simple fluid is defined as one for which the stress $\mathbf{T}$ at the position $\mathbf{x}$ and time t is specified within an indeterminate pressure p by the history of the relative right Cauchy-Green strain tensor for the material which is within an arbitrtrarily small neighborhood of $\mathbf{x}$ at time t,

$$T + p\ I = \frac{\mu_0}{s_0}\ \overset{\infty}{\underset{\sigma=0}{\mathbf{H}^+}}\ [\mathbf{C}_{(t)}(t - s_0\ \sigma)] \qquad (61)$$

Here we follow Truesdell's discussion of the dimensional indifference of the definition of a simple material (*6*, p. 65; *11*). The quantity $\overset{\infty}{\underset{\sigma=0}{\mathbf{H}^+}}$ is a dimensionally invariant tensor-valued functional, that is, an operator which maps tensor-valued functions into tensors. The constants μ_0 and s_0 are respectively a characteristic viscosity and characteristic time or natural time lapse of the fluid. The characteristic viscosity and characteristic time may be defined arbitrarily. A reasonable definition for the characteristic viscosity seems to be the viscosity observed in the limit as the rate of deformation approaches zero. The characteristic time of the fluid is discussed in detail elsewhere (*4*).

Equations (8) and (9) allow Equation (6) to be written for flow through a stationary porous structure as

$$\nabla(\overline{\mathcal{P} - p_0}) - \mathrm{div}\,\overline{(\mathbf{T} + p\ \mathbf{I})} + R\ \overline{\mathbf{v}} = 0 \qquad (62)$$

Let us introduce dimensionless variables as we did in the preceding section with the additional definition

$$\overline{(\mathbf{T} + p\ \mathbf{I})}^+ = \frac{s_0}{\mu_0}\ \overline{(\mathbf{T} + p\ \mathbf{I})} \qquad (63)$$

This allows us to write Equation (62) in a dimensionless form as

$$\frac{P}{L\ R\ v}\ \nabla^+(\overline{\mathcal{P} - p_0})^+ - \frac{\mu_0}{s_0\ R\ v\ L}\ \mathrm{div}(\overline{\mathbf{T} + p\ \mathbf{I}}) + \overline{\mathbf{v}}^+ = 0 \qquad (64)$$

In order to get a feeling for magnitude of the parameter multiplying the second term on the left of Equation (64), let us consider an example using some typical values for the various characteristic quantities:

μ_0/R = 250 millidarcies or 2.5×10^{-9} sq.cm.

$s_0 = 10^{-2}$ sec., v = 1 ft./day or 3.5×10^{-4} cm./sec.

$$\frac{\mu_0}{s_0\ R\ v\ L} = \frac{7 \times 10^{-4}}{L} \qquad (65)$$

A characteristic time of 10^{-2} sec. appears to be reasonable for some viscoelastic fluids (*12*, *13*). If we remember that L was introduced as a length characteristic of the gross geometry [previously (*1*, *2*), we used a length which was characteristic of the porous medium, such as the square root of the salt water permeability], this suggests that the second term on the left of Equation (60) may be neglected with respect to the third term to obtain

$$\nabla(\overline{\mathcal{P} - p_0}) + R\ \overline{\mathbf{v}} = 0 \qquad (66)$$

This is essentially the result suggested previously [*1*, equation (22)]. But the precautions noted in the previous section with respect to a similar argument for Newtonian fluids are applicable here as well.

VISCOELASTIC TUBE FLOW

By a previous argument employing the Buckingham-Pi theorem, we suggested that for a Noll simple fluid [*1*, equations (49) and (52)]

$$R = \frac{\mu_0}{L^2\ \kappa^+},\ \kappa^+ = \kappa^+\left[\frac{|\nabla(\overline{\mathcal{P} - p_0})|\ L\ s_0}{\mu_0}\right] \qquad (67)$$

Working with a capillary tube model for a porous medium, McKinley, Jahns, Harris, and Greenkorn (*14*) derived a result somewhat similar to Equations (66) and (67) for a more restricted class of fluid behavior. Their velocity and pressure are only vaguely defined as the result of their derivation. Experimental data (see Figure 1) suggest that, over a limited range of $|\nabla(\overline{\mathcal{P} - p_0})|$, a useful relation would be of the form

$$1/R = m \cdot |\nabla(\overline{\mathcal{P} - p_0})|^{n-1} \qquad (68)$$

or, in order to account for Newtonian behavior at very

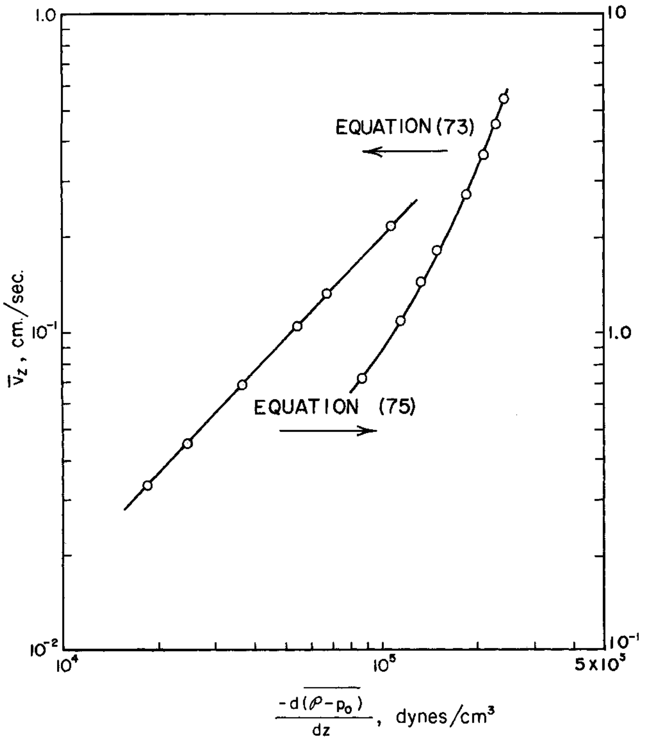

Fig. 1. Comparison of Equation (73) with data for 1.25% carboxymethylcellulose solution in water *(16)* ($\alpha = 7.14 \times 10^{-7}$ cm./sec.; $\beta = 5.49 \times 10^{-6}$ sq. cm./dyne; $\gamma = 3.5$). Comparison of Equation (75) with data for 18.5% Carbowax (polyethylene glycol, viscosity average molecular weight equal to 20,000) solution in water *(17)* ($m = 8.90 \times 10^{-6}$; $n = 1.07$).

low values of $|\nabla(\overline{\mathcal{P}-p_0})|$,

$$1/R = \alpha\,[1 + \{\beta\,|\nabla(\overline{\mathcal{P}-p_0})|\}^{\gamma-1}] \qquad (69)$$

Equations (68) and (69) are analogous to the familiar Power model and Ellis model of rheology respectively [*1*, equations (23) and (34)], but one should not necessarily expect any relation between, for example, the parameters appearing in Equation (69) and those appearing in the Ellis model. Equations (68) and (69) are empiricisms which may be helpful in correlating data for the flow of Noll simple fluids in porous media.

As an example of how one might use Equation (66), consider flow through a porous media bounded by an infinite impermeable cylindrical surface. We assume boundary conditions (43) and (44) apply and we require

$$\text{at} \quad r = r_0 + \epsilon: \quad \bar{v}_r = 0 \qquad (70)$$

We assume that the resistance coefficient can be described by Equation (69) and we ask whether there is a solution of the form of Equations (45). The equation of continuity, Equation (4), is satisfied by Equations (45); the three components of Equation (66) become

$$\frac{1}{R}\,\frac{\partial(\overline{\mathcal{P}-p_0})}{\partial z} + \bar{v}_z = 0 \qquad (71)$$

and

$$\frac{\partial(\overline{\mathcal{P}-p_0})}{\partial r} = \frac{\partial(\overline{\mathcal{P}-p_0})}{\partial \theta} = 0 \qquad (72)$$

With the help of Equations (69) and (72), Equation (71) can be written as

$$\bar{v}_z = -\alpha\,\frac{d(\overline{\mathcal{P}-p_0})}{dz}\left[1 + \left\{-\beta\,\frac{d(\overline{\mathcal{P}-p_0})}{dz}\right\}^{\gamma-1}\right] = \text{a constant} \qquad (73)$$

This equation implies that

$$\frac{d(\overline{\mathcal{P}-p_0})}{dz} = C = \text{a constant} \qquad (74)$$

Upon integration and application of boundary conditions (43) and (44), we find that C is again given by Equation (47)

For the same problem, if the resistance coefficient is described by Equation (68), we obtain

$$\bar{v}_z = m\left[-\frac{d(\overline{\mathcal{P}-p_0})}{dz}\right]^n \qquad (75)$$

Equations (47) and (74) are again applicable.

Using the method suggested by Ashare, Bird, and Lescarboura (*15*), we have fit Equation (73) to the data of Christopher (*16*) for a 1.25% carboxymethyl cellulose solution in water flowing through a bed 17.6 cm. long. Figure 1 indicates excellent agreement with the experimental data. Figure 1 also illustrates that Equation (75) represents very well the data of Sadowski (*17*) for a 18.5% Carbowax (polyethylene glycol, viscosity average molecular weight equal to 20,000) solution in water. In this way, we confirm the potential utility of Equations (68) and (69).

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NOTATION

$\mathbf{f}$ = external force per unit mass
$\mathbf{g}$ = force per unit volume which the fluid exerts on the pore walls contained within S beyond the hydrostatic force and beyond any force attributable to the ambient pressure
$\mathbf{I}$ = identity transformation
L = characteristic length of gross geometry
m, n = parameters appearing in Equation (68)
$\mathbf{n}$ = outwardly directed unit normal
p = pressure
p_0 = reference or ambient pressure
$\mathbf{p}$ = position vector in old frame
$\mathbf{p}_0$ = a fixed position in the old frame
$\mathcal{P}$ = local volume-averaged modified pressure; defined by Equation (7)
$\mathbf{Q}(t)$ = time-dependent orthogonal transformation
R = resistance coefficient for a nonoriented porous structure; defined by Equation (19)
s_0 = characteristic time or time lapse of an incompressible Noll simple fluid
S = arbitrary closed surface which is associated with every point in the porous medium
$S_{(s)}$ = closed bounding surface of $V_{(s)}$
S_e = portion of $S_{(s)}$ which coincides with S
S_f = portion of $S_{(s)}$ which coincides with the pore walls
t = time
$\mathbf{T}$ = stress tensor

$\bar{u}$ = local volume-averaged velocity of the solid; defined by Equation (11)
v = velocity
V = volume of S
$V_{(s)}$ = volume of pores containing fluid which are enclosed by the surface S

Greek letters

α, β, γ = parameters appearing in Equation (69)
ϵ = distance into an impermeable wall at which $\bar{\underset{\sim}{v}} = 0$
μ = viscosity
ρ = density
ϕ = external force potential; defined by Equation (5)
ψ = porosity

Special symbols

— = an average over the volume V of quantities associated with the fluid; see Equation (3)
* = a quantity associated with a new reference frame
+ = a dimensionless variable
$\wedge$ = indicates a vector product

LITERATURE CITED

1. Slattery, J. C., *AIChE J.*, 13, 8, 1066 (1967).
2. *Ibid.*, 14, 50 (1968).
3. *Ibid.*, 5, 831 (1965).
4. *Ibid.*, 14, 516 (1968).
5. Truesdell, C., "The Elements of Continuum Mechanics," Springer-Verlag, New York (1966).
6. ———, and W. Noll, "Handbuch der Physik," III/3, S. Flügge, ed., Springer-Verlag, Berlin (1965).
7. Spencer, A. J., and R. S. Rivlin, *Arch. Rational Mech. Analysis*, 4, 214 (1959/60).
8. Smith, G. F., *ibid.*, 18, 282 (1965).
9. Brinkman, H. C., *Appl. Sci. Res.*, A1, 27, 81 (1949).
10. Irving, J., and N. Mullineux, "Mathematics in Physics and Engineering," Academic Press, New York (1959).
11. Truesdell, C., *Phys. Fluids*, 7, 1134 (1964).
12. Shertzer, C. R., and A. B. Metzner, *Proc. 4th Int. Congr. Rheol*, Pt. 2, 603 (1965).
13. Ginn, R. F., and A. B. Metzner, *ibid.*, 583 (1965).
14. McKinley, R. M., H. O. Jahns, W. W. Harris, and R. A. Greenkorn, *AIChE J.*, 12, 17 (1966).
15. Ashare, Edward, R. B. Bird, and J. A. Lescarboura, *ibid.*, 11, 910 (1965).
16. Christopher, R. H., MS thesis, Univ. Rochester, New York (1965).
17. Sadowski, T. J., Ph.D. thesis, Univ. Wisconsin, Madison (1963).

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Combined Forced and Free Convective Diffusion in Vertical Semipermeable Parallel Plate Ducts

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The effect of natural convection on polarization and flow patterns in liquid phase convective diffusion in a vertical duct with semipermeable membrane walls has been investigated theoretically. It is found that at low flow rates, gravitational fields can play a significant role in distorting the velocity profiles and thereby they affect the transition from laminar to turbulent flow. Natural convection also significantly affects mass transfer rates and therefore the extent of polarization at low flow rates. Results are presented for both momentum and mass transfer in upward and downward flows for different wall Peclet numbers. The hydrodynamic stability of the system also has been investigated and critical values of the buoyancy parameters are reported. Also, these results enable one to estimate when natural convection may create errors in membrane testing systems.

The analysis and results are of practical interest in reverse osmosis and other membrane separation processes. The more productive the system, the more likely it will be that buoyancy effects are important.

In liquid-phase membrane separation processes, one is interested in understanding the fundamental transport processes occurring within the membrane structure and in studying the nonlinear diffusional effects on the feed solution side of the membrane. In particular, the phenomenon of polarization or accumulation of the rejected component at the membrane surface is significant since it limits the efficiency of such a process. Several papers, mostly of a theoretical nature have appeared in the literature, which have investigated methods of predicting the extent of polarization with various geometries under laminar and turbulent flow conditions (*3, 9* to *11, 20, 22*). It has been found that polarization can be a very important factor in the design of practical equipment.